ON THE TRANSPOSITIONAL RELATIONS IN ANALYTICAL MECHANICS OF NONHOLONOMIC SYSTEMS

(PERESTANOVOCHNYE SOOTNOSHENIIA V ANALITICHESKOI MEKHANIKE NEGOLONOMNYKH SISTEM)

PMM Vol.24, No.6, 1960, pp. 1013-1017

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(Received March 21, 1960)

In analytical mechanics of nonholonomic systems an essential role is played by so-called transpositional relations [1-10, 14, 21], which usually means the relations containing the operations $d\delta$ and δd, with d being differentiation with respect to time and δ being virtual variation. There are two types of transpositional relations corresponding to two points of view on the commutativeness of the operations d and δ in the presence of non-integrable kinematic constraints. According to one point of view (supported, for instance, by Volterra and Hamel), interchangeability of the operations d and δ exists in all the essential coordinates $q_1, \ldots, q_n$, independently of the fact that a system is holonomic or nonholonomic. According to the other point of view (Suslov, Levi-Civita and Amaldi), interchangeability of the operations d and δ exists only in the case of holonomic systems.

For nonholonomic systems, the relation

$$d\delta q_\tau - \delta d q_\tau = 0 \tag{0.1}$$

applies only for the generalized coordinates whose variations (compatible with nonholonomic constraints) may be considered as independent. The transpositional relations for the remaining coordinates are derived from the equations of nonholonomic constraints, and they prove to be different from (0.1). This second point of view has been generally recognized*, and its numerous adherents consider the first point of view as being erroneous [7-13]. Hamel remarked in [14] that the accusation of error

* Also in [23], the relations (0.1) are assumed not for all generalized coordinates.

is not irrefutable, and he proposed a method of utilization of the transpositional relations in which the first point of view does not lead to a contradiction. Hamel did not, however, justify his considerations.

The question of the transpositional relations was clarified in [15], where it was shown that, with a proper approach, both points of view are correct and are not contradictory, as had been alleged. It was explained that the contradiction was due to lack of definition of basic concepts, i.e. of the operations $d\delta$ and δd entering into the transpositional relations. In fact, the quantity $\dot{q}_r$ is determined only on the trajectory of motion, while the operation δ represents variations in the directions which are, in general, different from the direction of actual motion. It is thus necessary to determine from the very beginning the meaning of the operations d, δ, $d\delta$ and δd. We note that it is sufficient to define these operations on an arbitrary line $q_i = q_i(t)$ of the actual or kinematically admissible motion, and it should be done in relation to this given line. The definition of the operations d and δ is equivalent to the prescribing of vector fields corresponding to these operations.

1. Some definitions. In mechanics, the operation d represents differentiation with respect to time, and therefore is determined only for the points $q_i = q_i(t)$ of the trajectory of motion of the system. The correspondence can be established between the operation d and the field of vectors with the components $q_1 dt, \ldots, q_n dt$. An arbitrary operation from an infinite set is understood as the virtual variation δ, for which the corresponding vectors are virtual displacements of the system, i.e. all linear combinations of m linearly independent vectors $\mathbf{l}_1, \ldots, \mathbf{l}_m$ obtained in a certain way according to kinematic constraints of the system. Thus, the operation δ is determined at all points of configuration space.

It follows then that at the points of an arbitrary (actual or kinematically admissible) trajectory the operation $d\delta$ is determined, but the operation δd is not determined. Consequently, the definition of the operation d should be complemented in such a way that the operation δd becomes meaningful. It is important to note that the operations d and δ may be defined arbitrarily beyond the trajectory of motion $q_i = q_i(t)$, but on this trajectory they should coincide with the operations of differentiation with respect to time and virtual variation, respectively, in order to maintain the validity of the equations of d'Alembert and Lagrange.

There is an infinite number of ways of complementing the definitions of the operations d and δ while maintaining their meanings on the trajectory of motion. Here, the following way will be outlined. We introduce in the vicinity of the considered motion $q_i = q_i(t)$ a system of curvilinear coordinates $q_i = q_i(u_1, u_2, \ldots, u_n)$ in such a way that the lines $u_2 = u_3 = \ldots = u_n = 0$ coincide with the trajectories of motion; the parameter u_1 on this trajectory is identical with time t and the planes tangent to the surfaces $u_{m+1} = \ldots = u_n = 0$ at the points $u_2 = u_3 = \ldots = u_n = 0$ are the planes of virtual displacements of the system.

We assume, to be specific, that the equations of kinematic constraints are linear and nonholonomic of the type (1.2). In the surrounding of the line $q_i = q_i(t)$ we define

$$\begin{aligned} dq_\tau &= \frac{\partial q_\tau}{\partial u_1} du_1, \qquad \delta q_\tau = \frac{\partial q_\tau}{\partial u_r} \delta u_r \qquad (\tau = 1, \ldots, m+k; r = 1, \ldots, m) \\ dq_\tau &= a_{\tau s}\, dq_s, \qquad \delta q_\tau = a_{\tau s}\, \delta q_s \qquad (\tau = m+k+1, \ldots, n; s = 1, \ldots, m) \end{aligned} \tag{1.1}$$

Here and in the following, summations are to be carried on with respect to indices repeated twice, m is the number of degrees of freedom, k is a constant integer $(0 \leqslant k \leqslant n - m)$, and the indices assume the values

$$r, s, l = 1, \ldots, m; \quad i = 1, \ldots, n; \quad j = m+1, \ldots, n; \quad \rho = m+1, \ldots, m+k$$

$$\alpha, \beta, \lambda, \mu, \nu = 1, \ldots, m+k; \quad \sigma = m+k+1,\ m+k+2, \ldots, n$$

For $u_2 = u_3 = \ldots = u_n = 0$, the operation d coincides with differentiation with respect to time, and the operation δ coincides with virtual variation. According to the introduced definition, the interchangeability of the operations d and δ exists only for $r = 1, \ldots, m+k$; the displacements dq_r and δq_r are compatible with the constraints for all the values of r, except $r = m+1, \ldots, m+k$.

For a nonholonomic system with the constraints

$$\dot{q}_j = a_{js}\dot{q}_s \tag{1.2}$$

we introduce quasi-coordinates $\pi_1, \ldots, \pi_{m+k}$ by means of the relations

$$\dot{\pi}_r = a_{rs}\dot{q}_s, \qquad \dot{\pi}_\rho = a_{\rho s}\dot{q}_s - \dot{q}_\rho \tag{1.3}$$

According to (1.1) to (1.3) we obtain the following relations:

$$d\delta q_\lambda - \delta dq_\lambda = 0, \quad d\delta\pi_\lambda - \delta d\pi_\lambda = \gamma_{\nu\lambda\mu}\, d\pi_\mu \delta\pi_\nu, \quad d\delta q_\sigma - \delta dq_\sigma = B_{rs}^{\sigma} dq_r \delta q_s \tag{1.4}$$

where

$$\gamma_{\nu\lambda\mu} = b_{\alpha\nu} b_{\beta\mu} \left(\frac{\partial a_{\lambda\alpha}}{\partial q_\beta} - \frac{\partial a_{\lambda\beta}}{\partial q_\alpha} \right), \qquad b_{\alpha\lambda} a_{\lambda\beta} = \delta_{\alpha\beta} \tag{1.5}$$

($\delta_{\alpha\beta}$ is Kronecker's symbol)

$$B_{rs}^{\sigma} = \frac{\partial a_{\sigma s}}{\partial q_r} + \frac{\partial a_{\sigma s}}{\partial q_j} a_{jr} - \frac{\partial a_{\sigma r}}{\partial q_s} - \frac{\partial a_{\sigma r}}{\partial q_j} a_{js} \tag{1.6}$$

Two points of view mentioned above correspond to $k = n - m$ and $k = 0$.

2. Different forms of the equations of motion of nonholonomic systems and transpositional relations. Using relations (1.4), it is possible to obtain the equations of motion in the form from which, in particular, the equations in quasi-coordinates of Boltzmann and Hamel, or the equations in true coordinates of Voronets and Chaplygin, can be obtained.

We transform the equation of d'Alembert and Lagrange to the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\delta q_i\right) - \delta T + \frac{\partial T}{\partial q_i}\left(\delta q_i - \frac{d}{dt}\delta q_i\right) = Q_i \delta q_i \tag{2.1}$$

Substituting the first and the third of relations (1.4), we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\delta q_i\right) - \delta T - \frac{\partial T}{\partial \dot{q}_\sigma} B_{rs}^{\sigma}\dot{q}_r \delta q_s = Q_i \delta q_i \tag{2.2}$$

According to (1.2 and (1.3), the variations of coordinates and quasi-coordinates are connected by the relations

$$\delta q_\lambda = b_{\lambda\nu}\delta\pi_\nu, \quad \delta q_\sigma = a_{\sigma s}\delta q_s = a_{\sigma s} b_{s\nu}\delta\pi_\nu, \quad (b_{\lambda\nu}a_{\nu\mu} = \delta_{\lambda\mu}) \tag{2.3}$$

We introduce the function

$$T^*(q_i, \dot{\pi}_\nu) = T(q_i, b_{\lambda\nu}\dot{\pi}_\nu, a_{\sigma s} b_{s\nu}\dot{\pi}_\nu) \tag{2.4}$$

which is obtained from the kinetic energy $T(q_i, \dot{q}_i)$ by eliminating the generalized velocities $\dot{q}_{m+k+1}, \dot{q}_{m+k+2}, \ldots, \dot{q}_n$ with the use of (1.2) and exchanging $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{m+k}$ for $\dot{\pi}_1, \dot{\pi}_2, \ldots, \dot{\pi}_{m+k}$ with the use of (1.3). Passing to quasi-coordinates in (2.2), we have

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{\pi}_\nu}\delta\pi_\nu\right) - \delta T^* - \frac{\partial T}{\partial \dot{q}_\sigma} B_{rs}^{\sigma} q_r b_{s\nu}\delta\pi_\nu = \Pi_\nu \delta\pi_\nu$$

or, after a transformation with the use of the second of the equations (1.4)

$$\left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_\nu} - \frac{\partial T^*}{\partial q_\lambda} b_{\lambda\nu} - \frac{\partial T^*}{\partial q_\sigma} a_{\sigma s} b_{s\nu} + \gamma_{\nu\lambda\mu}\frac{\partial T^*}{\partial \dot{\pi}_\lambda}\dot{\pi}_\mu - \frac{\partial T}{\partial \dot{q}_\sigma} B_{rs}^{\sigma}\dot{q}_r b_{s\nu} - \Pi_\nu\right)\delta\pi_\nu = 0$$

According to the conditions of nonholonomic constraints, it follows that $\delta\pi_{m+1} = \delta\pi_{m+2} = \ldots = \delta\pi_{m+k} = 0$. Since the variations $\delta\pi_l$ are independent, the remaining sum results in m equations of the type

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{\pi}_l} - \frac{\partial T^*}{\partial \pi_l} + \gamma_{l\lambda\mu}\frac{\partial T^*}{\partial \dot{\pi}_\lambda}\dot{\pi}_\mu - B_{rs}{}^\sigma \frac{\partial T}{\partial \dot{q}_\sigma}\dot{q}_r b_{sl} = \Pi_l \tag{2.5}$$

where

$$\frac{\partial T^*}{\partial \pi_l} = \frac{\partial T^*}{\partial q_\lambda} b_{\lambda l} + \frac{\partial T}{\partial q_\sigma} a_{\sigma s} b_{sl}. \tag{2.6}$$

The obtained equations (2.5) represent a compromise between the equations of Boltzmann and Hamel in quasi-coordinates and the equations of Chaplygin and Voronets.

In fact, for $k = n - m$ the components

$$B_{rs}^\sigma \frac{\partial T}{\partial \dot{q}_\sigma} q_r b_{sl}$$

vanish and Equations (2.5) coincide with the equations of Boltzmann and Hamel [4, 17].

For $k = 0$ Equations (2.5) become the equations of Chaplygin and Voronets. They are written in true coordinates and, therefore, it is necessary to substitute $a_{sl} = b_{sl} = \delta_{sl}$ in relations (2.5). Hence, and from (1.5), it follows that $\gamma_{l\lambda\mu} = 0$.

In addition, for $k = 0$ the function T^* becomes, according to (2.4), the function $T^\circ = T^\circ(q_i, \dot{q}_i)$, which is commonly used in deriving the equations of Voronets. In this way the equations

$$\frac{d}{dt}\frac{\partial T^\circ}{\partial \dot{q}_l} - \frac{\partial T^\circ}{\partial q_j} a_{jl} - B_{rl}^j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_r = \Pi_l \tag{2.7}$$

are obtained, which are identical with the equations of Voronets [1].

For nonholonomic systems of Chaplygin, they coincide with the known equations of Chaplygin [18].

3. Different forms of Hamilton's principle and the transpositional relations. According to Hamilton's principle, the actual motion of a dynamical system satisfies the equation

$$\int_{t_0}^{t_1} (\delta T + \delta A)\, dt = 0 \tag{3.1}$$

where δT is the virtual variation of the kinetic energy of the system and δA is the virtual work of applied forces. Usually, this principle is formulated for holonomic systems. Up to now, there is no unique point of view on its applicability for nonholonomic systems. Some authors (for instance, Appell [20]) consider that Hamilton's principle (3.1) should not be applied to nonholonomic systems. Others (for instance, Hamel [22]) maintain the opposite point of view (Chaplygin pointed out the applicability of Hamilton's principle to nonholonomic systems which admit a reducing factor).

But, at the same time, a general approach to this problem is possible, since the applicability of Hamilton's principle is closely related to the question of the transpositional relations.

In fact, let us integrate Equation (2.1) in the interval from the initial state to the final state of the system. With the condition that the variations are zero at the ends of this interval, the expression

$$\int_{t_0}^{t_1}\left[\delta T+\delta A+\frac{\partial T}{\partial \dot{q}_i}\left(\frac{d}{dt}\,\delta q_i-\delta\dot{q}_i\right)\right]dt=0 \tag{3.2}$$

is obtained, which may be considered as a general formulation of Hamilton's principle. It follows from (3.2) that a particular form of the principle depends on the assumption of certain transpositional relations. The form (3.1) corresponds to the relations (1.4) with $k = n - m$, and leads to the equations of Boltzmann and Hamel, according to the previous considerations.

Note. Voronets [1] proposed a method of derivation of the equations of motion for nonholonomic conservative systems starting from the expression (using the notation of this paper)

$$\int_{t_0}^{t_1}[\delta(\widetilde{T}+U)+\frac{\partial T}{\partial \dot{q}_j}\,\delta(\dot{q}_j-a_{js}\dot{q}_s)]\,dt=0 \tag{3.3}$$

with the condition of interchangeability of the operations d and δ for all essential coordinates. In spite of the apparent differences between (3.3) and (3.1), both these forms result from the same transpositional relations and, therefore, one expression can be reduced to another. In fact, assuming the true coordinates as first m quasi-coordinates, we have

$$\delta T^*=\frac{\partial T^*}{\partial\dot{\pi}_s}\,\delta\dot{\pi}_s+\frac{\partial T^*}{\partial\dot{\pi}_j}\,\delta\dot{\pi}_j+\frac{\partial T^*}{\partial q_i}\,\delta q_i=\delta\,\widetilde{T}+\frac{\partial T}{\partial\dot{q}_j}\,\delta(\dot{q}_j-a_{js}\dot{q}_s)$$

Substituting this into (3.1), we obtain expression (3.3) of Voronets.

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Translated by M. P. B.